MATH 8702 TOPICS IN APPLIED MATH: BIFURCATION THEORY

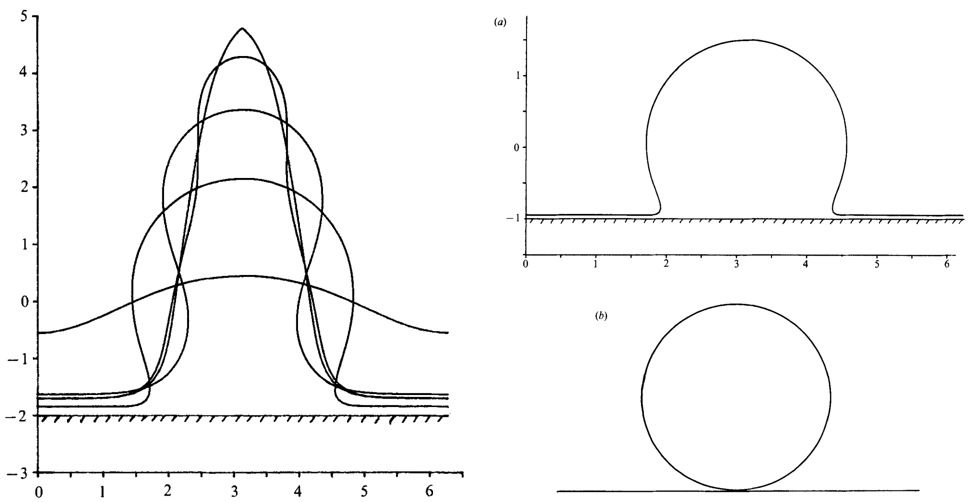

Figure 1. Families of traveling water waves numerically computed by Da Silva and Peregrine [1]. Left: these solutions limit to an extreme wave with a corner at its crest; Right: Along another branch, they find *overhanging waves* that appear to limit to a solution wave where the water region separates into a rigidly rotating ball superposed on a flat line.

Many problems in mathematics boil down to solving an abstract operator equation of the form

$$F(u, \lambda) = 0,$$

where u is an unknown lying in some Banach space, and $\lambda \in \mathbb{R}$ is a parameter. Any PDE can be expressed this way, with the corresponding F being a mapping between infinite-dimensional Banach spaces.

Often, these types of equations have a "trivial solution" or a family of trivial solutions. For example, suppose that

$$F(0, \lambda_0) = 0 \qquad \text{or} \qquad F(0, \lambda) = 0, \text{ for all } \lambda \in \mathbb{R}.$$

Bifurcation theory offers a robust strategy for finding new, nontrivial solutions. First, we look for a curve of nontrivial solutions that emanates from a trivial solution, and then we continue the curve as far possible. The initial stage is called *local bifurcation*, while the more subtle step of extending the curve is called *global bifurcation*. Together, they comprise an extremely powerful tool in nonlinear functional analysis. Notice that the global curve can wander far from the trivial solution, and so we can hope to see very wild behavior as we follow along it.

Perhaps the most famous application of global bifurcation theory is the celebrated resolution of the Stokes Conjecture. In 1880, Stokes formally constructed a family of surface water waves (that is, solutions of the 2-d irrotational and incompressible free boundary Euler problem) that are periodic. He claimed that the tallest such wave, called the *extreme wave*, must have a 120° corner at each crest. It took over 100 years, and a remarkable bifurcation theoretic proof, to rigorously confirm this fact. A major subject of current research is the existence of even more dramatic solutions, such as the overhanging waves pictured above.

Course description. The objective of this course is to give an introduction to bifurcation theoretic methods, particularly as they apply to elliptic PDEs. We will study both local and global theory, but the emphasis will be on the global side. We will give a thorough treatment of the analytic global bifurcation theory of Dancer, and its refinement by Buffoni and Toland. Mathematically, this machinery exploits deep facts from functional analysis and properties of (real and complex) analytic varieties. As a concrete application, we will apply these techniques to find large-amplitude traveling water waves.

Textbook. The main references for the course will be

> *Bifurcation Theory: An Introduction with Applications to PDEs* by H. Kielhöfer
> *Analytic Theory of Global Bifurcation* by J. F. Toland and B. Buffoni.

Prerequisites. A year of graduate analysis. Prior exposure to the material in PDE I is ideal but not required.

References

[1] A. F. Teles da Silva and D. H. Peregrine, *Steep, steady surface waves on water of finite depth with constant vorticity*, J. Fluid Mech., 195 (1988), pp. 281–302.